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The Leontief Model and Economic Theory

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Abstract: This article examines the relationship between classical, marginalist and Keynesian economics and the Leontief model and shows how the analysis of productive and distributional interdependencies may provide an appropriate conceptual framework for comparing the different analytical approaches.

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1. Introduction

The aim of this article is to analyse the relationship between classical, marginalist and Keynesian economics and the Leontief model, and to show how the analysis of productive and distributional interdependencies may provide an appropriate conceptual framework for comparing the various schools of economic thought and developing their premises to their logical conclusions.

All comparisons between different schools of thought are difficult because economic relationships are complex and each school deals with this complexity in different ways.

Classical and marginalist economists hold that full employment and the full utilisation of capital will be achieved if the economy is free to work untrammelled by restrictions, while Keynesian economists deny the existence of automatic and efficient mechanisms for the equilibrium of the factor market. These opposing conclusions are linked to the acceptance or otherwise of Say's Law (*la loi des débouchés*), according to which ‘supply creates its own demand’ [Say, 1817].

Moreover, classical economists claim that, on the basis of the wage fund theory (the fixed quantity of goods available in a society at any given time to advance wages to workers involved in the production process), the imperfect working of Say's Law can be corrected by a reduction in real wages. For their part, Keynesian economists believe that a reduction in wages causes a fall in the demand for consumer goods, but has no significant substitution effect between labour and capital [Pasinetti, 1981; Baranzini and Scazzieri, 1990; Morishima, 1990].

At the analytic-conceptual level, the different schools of economic thought hold opposing views, considering economics either as a ‘real science’, situated in a historical context and concerned with the organisation of society, or as a ‘formal science’ unrelated to time and based on rational principles [Schumpeter, 1954; Carnap, 1935]. In these terms, any comparison seems impossible.

The Leontief model has elements in common with these different approaches and can thus serve as an *algorithm for translation,* allowing us to move from one conceptual frame of reference to another. Sectoral interdependence is able to represent, at least in the spatial dimension, the complex organisation of a society founded on the division of labour, the distribution of income, the accumulation of capital and the exchange of goods on the market. At the same time, sectoral interdependence offers a powerful analytical tool for the solution of problems of equilibrium and of maximum and minimum typical of rational calculus using the formal procedures of the algebra of matrices.

In his introduction to *The Structure of American Economy 1919-1929*, the first systematic presentation of what was to become input-output analysis, Leontief described his work as ‘an attempt to apply the economic theory of general equilibrium - or better, general interdependence - to an empirical study of interrelations among the different parts of a national economy as revealed through covariations of prices, outputs, investments, and incomes’ [Leontief, 1941].

Leontief's original idea turned out to be extremely fruitful and the input-output approach extended its fields of application beyond the limits of strictly defined economic processes. The majority of empirical applications, however, were limited to the use of the ‘open’ version of the model, which does not allow a complete solution of problems related to the relationship between the productive part of the economic system and those parts concerned with consumption and accumulation [Miller and Blair, 1985].

If we wish the input-output approach to represent the general interdependence of an economic system, we need to ‘close’ the model, that is to make the levels of consumption and investment endogenous or, in dual terms, to make the levels of wages and profits endogenous [Costa and Marangoni, 1995].

In order to close the model we need to introduce new assumptions alongside the basic hypothesis of the input-output model, which is that of constant coefficients of production. It is at this stage that we can take into consideration the different ways in which the various analytical approaches believe economic systems to work. The acceptance of one assumption rather than another leads to a particular 'closure' of the input-output model, which takes on classical, marginalist or Keynesian characteristics, while remaining a general frame of reference for representing an economic system and reasoning about it.

The article has been structured as follows. Section 2 presents the input-output model as a homogeneous system of linear equations. This formulation allows us to bring degrees of freedom about and consequently to introduce assumptions typical of the different analytical approaches. The homogeneous system also leads to a variety of 'Leontief inverses'. Sections 3, 4 and 5 discuss the marginalist, Keynesian and classical 'readings' of the Leontief model, each with its own distinctive features. Finally, our concluding remarks highlight the 'neutrality' of the model with regard to the various methods of reasoning, but also its substantially classical characteristics.

2. The input-output model and the variety of Leontief inverses

Let us consider an economic system made up of two productive sectors. The flow of goods between the sectors can be schematised by the following input-output table:

$$
\begin{array}{|cc|c|c|}
\hline
p_1x_{11} & p_1x_{12} & p_1d_1 & p_1x_1 \\
p_2x_{21} & p_2x_{22} & p_2d_2 & p_2x_2 \\
\hline
V_1 & V_2 & & V \\
\hline
p_1x_1 & p_2x_2 & D & \\
\hline
\end{array}
\;.
$$

The significance of the symbols is as follows $(i, j = 1,2)$: p_i price of commodity i, x_{ij} quantity of commodity i used as input by sector j; d_i quantity of commodity i destined for the final demand (consumption and investments), x_i output of sector i, V_i value added of sector i (wages and profits), V total value added, D total final demand.

Let us introduce the hypothesis of a constant relationship between input and output and define the coefficients of production:

$$a_{ij} = \frac{x_{ij}}{x_j}\,.$$

The input-output table can then be written in the form:

$$
\begin{array}{|cc|c|c|}
\hline
p_1a_{11}x_1 & p_1a_{12}x_2 & p_1d_1 & p_1x_1 \\
p_2a_{21}x_1 & p_2a_{22}x_2 & p_2d_2 & p_2x_2 \\
\hline
V_1 & V_2 & & V \\
\hline
p_1x_1 & p_2x_2 & D & \\
\hline
\end{array}
\;.
$$

If we consider the row and column totals, the following relations hold:

$$x_1 = a_{11}x_1 + a_{12}x_2 + d_1$$
$$x_2 = a_{21}x_1 + a_{22}x_2 + d_2$$

$$p_1 = a_{11}p_1 + a_{21}p_2 + v_1$$
$$p_2 = a_{12}p_1 + a_{22}p_2 + v_2$$

where v_i indicates the value added of sector *i* per unit of output.

In the traditional open model, d_i and v_i are considered as given exogenous variables, while x_i e p_i are the endogenous variables to be determined.

If we define the vectors and the matrices:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

the model can be written in compact form as:

$$\begin{matrix} \mathbf{x} = \mathbf{Ax} + \mathbf{d} \\ \mathbf{p} = \mathbf{pA} + \mathbf{v} \end{matrix} \qquad \text{or} \qquad \begin{matrix} (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{d} \\ \mathbf{p}(\mathbf{I} - \mathbf{A}) = \mathbf{v} \end{matrix}$$

and the solution is:

$$\begin{matrix} \mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d} \\ \mathbf{p} = \mathbf{v}(\mathbf{I} - \mathbf{A})^{-1} \end{matrix}.$$

The ‘Leontief inverse’ $(\mathbf{I} - \mathbf{A})^{-1}$ is the operator which allows us to solve the two classic problems faced by the input-output model: the search for vector **x** of sectoral production capable of satisfying the final demand **d** and the search for vector **p** of sectoral prices consistent with a given vector of sectoral coefficients of value added **v**.

The ‘Leontief inverse’ is, however, only one of the ‘inverses’ which can be used for the solution of the open model. The input-output model can be converted into a homogeneous system of linear equations which has more degrees of freedom than the traditional model [Costa and Marangoni, 1995].

The quantity and price equations can be written as:

$$x_1 - a_{11}x_1 - a_{12}x_2 - d_1 = 0$$
$$x_2 - a_{21}x_1 - a_{22}x_2 - d_2 = 0$$
$$p_1 - a_{11}p_1 - a_{21}p_2 - v_1 = 0$$
$$p_2 = a_{12}p_1 - a_{22}p_2 - v_2 = 0$$

or, in matrix form, as:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -1 & 0 \\ -a_{21} & 1-a_{22} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ d_1 \\ d_2 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1-a_{11} & -a_{21} & -1 & 0 \\ -a_{12} & 1-a_{22} & 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ v_1 \\ v_2 \end{bmatrix} = \mathbf{0}.$$

If we set:

$$\mathbf{H} = \begin{bmatrix} 1-a_{11} & -a_{12} & -1 & 0 \\ -a_{21} & 1-a_{22} & 0 & -1 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} x_1 \\ x_2 \\ d_1 \\ d_2 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{s}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

the quantity model may be written as follows:

$$\mathbf{Hs} = \mathbf{0}$$

or:

$$\mathbf{H}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{s}_2 = \mathbf{0}$$

and may be solved for $\mathbf{s}_1$:

$$\mathbf{s}_1 = -(\mathbf{H}_1)^{-1}\mathbf{H}_2\mathbf{s}_2 .$$

In this case the final demands for the two sectors are considered to be the exogenous variables whose value is given, while the productions of the two sectors are the unknown endogenous variables to be determined. However, the roles of exogenous variables and endogenous variables can be exchanged: we can, for example, consider the production x_1 of the first sector as given, and aim to determine the final demand necessary to absorb this production; or even consider productions x_1 e x_2 of both sectors as given and aim to determine the levels of final demand necessary to absorb these productions.

From a formal point of view, it will suffice to write, in the first case:

$$\mathbf{H} = \begin{bmatrix} -1 & -a_{12} & 1-a_{11} & 0 \\ 0 & 1-a_{22} & -a_{21} & -1 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} d_1 \\ x_2 \\ x_1 \\ d_2 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} -1 & -a_{12} \\ 0 & 1-a_{22} \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} 1-a_{11} & 0 \\ -a_{21} & -1 \end{bmatrix} \quad \mathbf{s}_1 = \begin{bmatrix} d_1 \\ x_2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} x_1 \\ d_2 \end{bmatrix}$$

and, in the second case:

$$\mathbf{H} = \begin{bmatrix} -1 & 0 & 1-a_{11} & -a_{12} \\ 0 & -1 & -a_{21} & 1-a_{22} \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} d_1 \\ d_2 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix} \quad \mathbf{s}_1 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In both cases the solution is:

$$\mathbf{s}_1 = -(\mathbf{H}_1)^{-1}\mathbf{H}_2\mathbf{s}_2 .$$

This procedure is not a purely mathematical-formal exercise, but is based on a precise philosophy of the role played by demand and supply in an economic system. If, as in Keynesian economics, we recognise that demand has an autonomous and driving role, it becomes the exogenous control variable and levels of production align themselves accordingly. On the other hand, when, as in classical economics, emphasis is placed on the supply of goods and services, or, as in the marginalist model, the emphasis is on the alternative use of scarce resources, it is the levels of production which determine the performance of the whole economic system and create sufficient levels of demand.

It is worth pointing out that the Leontief model is limited to determining the theoretical levels of production necessary to satisfy a certain final demand or, in the version presented here, the theoretical levels of final demand necessary to absorb a certain production [Pasinetti, 1977]. Indeed, the model does not guarantee that the economic system will perform accordingly: shortage of labour and capital factors may cause production to be insufficient to satisfy a certain level of demand, and the imperfections of the mechanisms of income distribution may prevent the potential demand from becoming effective demand.

Turning to the system of prices, we can, using an analogous notation, write:

$$\mathbf{H} = \begin{bmatrix} 1-a_{11} & -a_{21} & -1 & 0 \\ -a_{12} & 1-a_{22} & 0 & -1 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} p_1 \\ p_2 \\ v_1 \\ v_2 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} 1-a_{11} & -a_{21} \\ -a_{12} & 1-a_{22} \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{s}_1 = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

hence:

$$\mathbf{Hs} = \mathbf{0}$$

or:

$$\mathbf{H}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{s}_2 = \mathbf{0}\,.$$

If we solve it for $\mathbf{s}_1$ we have:

$$\mathbf{s}_1 = -(\mathbf{H}_1)^{-1}\mathbf{H}_2\mathbf{s}_2\,.$$

The system determines prices compatible with given levels of value added; that is, prices which enable companies to pay certain monetary wages for work and to reach satisfactory profit levels. In this case, too, it is possible to assume the inversion of roles between exogenous variables (value added per unit of output) and endogenous variables (prices). The inversion of roles corresponds to the hypothesis (certainly drastic in this context) of fixed prices, to which wages and profits must conform.

Thus we would have:

$$\mathbf{H} = \begin{bmatrix} -1 & 0 & 1-a_{11} & -a_{21} \\ 0 & -1 & -a_{12} & 1-a_{22} \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} v_1 \\ v_2 \\ p_1 \\ p_2 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} 1-a_{11} & -a_{21} \\ -a_{12} & 1-a_{22} \end{bmatrix} \quad \mathbf{s}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

with the identical formal solution:

$$\mathbf{s}_1 = -(\mathbf{H}_1)^{-1}\mathbf{H}_2\mathbf{s}_2\,.$$

3. Endogenous final demand and the marginalist theory

Let us return to the simple economic system with two productive sectors as presented above.

And let us suppose that, of the two productive sectors, the first produces consumer goods, the second capital goods. We will divide income earners into two classes: the workers, who receive wages, and the capitalists, who receive profits. The workers spend their total income on the purchase of consumer goods, while the capitalists use all of their profits to purchase capital goods. These two situations, but particularly the second, imply the full acceptance, typical of the marginalist school, of Say's Law, in that both of them guarantee the complete absorption of the supply by the demand. Again, both situations, but particularly the first, imply the introduction of a multiplier mechanism which further boosts the production of consumer goods and investment as and when income rises [Miyazawa, 1976].

Each worker receives a real wage *w*, or given share of consumer goods. The monetary wage is equal to $p_1 w$. In order to produce consumer and capital goods it is necessary to employ labour and capital in fixed proportions: l_1, l_2, k_1 and k_2 indicate the respective technical coefficients. Profits are calculated on capital employed, according to a given rate of profit *r*. The total number of workers employed is equal to *L*, while the total quantity of capital employed is equal to *K*.

The following input-output table shows the flow of goods between the two sectors, the distribution of income between the two classes, and the use of income for the purchase of goods:

$p_1a_{11}x_1$	$p_1a_{12}x_2$	p_1wL		p_1x_1
$p_2a_{21}x_1$	$p_2a_{22}x_2$		p_2rK	p_2x_2
$p_1wl_1x_1$	$p_1wl_2x_2$			p_1wL
$p_2rk_1x_1$	$p_2rk_2x_2$			p_2rK
p_1x_1	p_2x_2	p_1wL	p_2rK	

The model which results from this table is:

- ‘closed’ with respect to consumption, which depends on the workers' income (in fact it coincides with this income, since we have set the propensity to consume as equal to one);

- ‘closed’ with respect to investments, which depend on and coincide with the capitalists' income;

- ‘closed’ with respect to wages, in that the monetary wages p_1w adapt automatically to changes in the prices of consumer goods in order to keep the workers' purchasing power the same.;

- ‘closed’ with respect to profits, which are calculated on the basis of capital whose value is automatically revalued when prices vary.

Hence we have:

$$x_1 = a_{11}x_1 + a_{12}x_2 + wL$$
$$x_2 = a_{21}x_1 + a_{22}x_2 + rK$$
$$L = l_1x_1 + l_2x_2$$
$$K = k_1x_1 + k_2x_2$$

$$p_1 = a_{11}p_1 + a_{21}p_2 + wl_1p_1 + rk_1p_2$$
$$p_2 = a_{12}p_1 + a_{22}p_2 + wl_2p_1 + rk_2p_2 \, .$$

If we use matrix notation, we can write the input-output model as a homogeneous system of linear equations:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -1 & 0 & 0 & 0 \\ -a_{21} & 1-a_{22} & 0 & -1 & 0 & 0 \\ -l_1 & -l_2 & 0 & 0 & 1 & 0 \\ -k_1 & -k_2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ wL \\ rK \\ L \\ K \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1-a_{11} & -a_{21} & -l_1 & -k_1 \\ -a_{12} & 1-a_{22} & -l_2 & -k_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ wp_1 \\ rp_2 \end{bmatrix} = \mathbf{0}\,.$$

The first thing we observe is that the quantity system and the price system are now linked by real wages w and the profit rate r. The solution of the two systems can be reached in two stages, by solving first the quantity system and then the price system.

The quantity system is a homogeneous linear system of four equations in six unknowns, which can be solved after the value of two unknowns is fixed. The solution of the system determines, directly or indirectly, the value of all six unknowns, including w and r.

If we replace the w and r values in the price system, this becomes a homogeneous linear system of two equations in two unknowns. One equation is linearly dependent on the other, so it may be eliminated and it is necessary to fix the value of one of the two variables p_1 or p_2. This operation is perfectly legitimate from the economic point of view in that, in every economic system, prices are always relative and depend on the commodity chosen as *numéraire*.

But let us return to the quantity system. Autonomous final demand does not appear among the variables, because of Say's Law, which guarantees complete absorption of production. If we wish to be consistent with the marginalist model, the system must guarantee full employment of all the factors. We can achieve this result by making use of the two degrees of freedom which the system offers: in other words, we can fix the values of L and K at the level of full employment.

The quantity system becomes:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -L & 0 \\ -a_{21} & 1-a_{22} & 0 & -K \\ l_1 & l_2 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L \\ K \end{bmatrix}$$

with the solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ w \\ r \end{bmatrix} = \begin{bmatrix} 1-a_{11} & -a_{12} & -L & 0 \\ -a_{21} & 1-a_{22} & 0 & -K \\ l_1 & l_2 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ L \\ K \end{bmatrix}.$$

If the solution makes economic sense, that is, if the variables take on non-negative values, the system determines, among other things, the value of w which, at this point, we are entitled to consider compatible with full employment. This situation is likewise consistent with the marginalist theory, which allows for wage flexibility.

If we place the w and r values in the price system we get:

$$\begin{bmatrix} 1-a_{11}-wl_1 & -a_{21}-rk_1 \\ -a_{12}-wl_2 & 1-a_{22}-rk_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \mathbf{0}.$$

The system is solved, as we have said, by considering the price of the commodity chosen as *numéraire* to be equal to one.

4. The open model with respect to investments: autonomous investments and the Keynesian equilibrium of underemployment

The automatic mechanisms which keep the labour and capital markets in equilibrium do not work if the investment decisions of companies are independent of profit levels (or, since only capitalists save, of level of savings).

The investments I are an autonomous component of the final demand. From the modified table:

$p_1 a_{11} x_1$	$p_1 a_{12} x_2$	$p_1 wL$		$p_1 x_1$
$p_2 a_{21} x_1$	$p_2 a_{22} x_2$		$p_2 I$	$p_2 x_2$
$p_1 w l_1 x_1$	$p_1 w l_2 x_2$			$p_1 wL$
$p_2 r k_1 x_1$	$p_2 r k_2 x_2$			$p_2 rK$
$p_1 x_1$	$p_2 x_2$	$p_1 wL$	$p_2 I$	

we can write the following:

$$x_1 = a_{11} x_1 + a_{12} x_2 + wL$$
$$x_2 = a_{21} x_1 + a_{22} x_2 + I$$
$$L = l_1 x_1 + l_2 x_2$$
$$K = k_1 x_1 + k_2 x_2$$

$$\begin{aligned} p_1 &= a_{11} p_1 + a_{21} p_2 + w l_1 p_1 + r k_1 p_2 \\ p_2 &= a_{12} p_1 + a_{22} p_2 + w l_2 p_1 + r k_2 p_2 \end{aligned}.$$

The input-output model may be set in the form of a homogeneous system of linear equations:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -1 & 0 & 0 & 0 \\ -a_{21} & 1-a_{22} & 0 & -1 & 0 & 0 \\ -l_1 & -l_2 & 0 & 0 & 1 & 0 \\ -k_1 & -k_2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ wL \\ I \\ L \\ K \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1-a_{11} & -a_{21} & -l_1 & -k_1 \\ -a_{12} & 1-a_{22} & -l_2 & -k_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ wp_1 \\ rp_2 \end{bmatrix} = \mathbf{0}\,.$$

In comparison with the marginalist model, the Keynesian model is now ‘open’ with respect to investments.

The quantity system, with its four equations and six unknowns, presents only one degree of freedom, since the value of I is given. We may now fix the value of the second variable, real wages w. This choice incorporates the Keynesian concept of the non-flexibility of wages: according to this model, a reduction in wages does not lead to the substitution of factors of production, but simply depresses consumption.

The choices we have made enable us to write the quantity system in the form:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -w & 0 \\ -a_{21} & 1-a_{22} & 0 & 0 \\ -l_1 & -l_2 & 1 & 0 \\ -k_1 & -k_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ L \\ K \end{bmatrix} = \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}.$$

The system can be solved thus:

$$\begin{bmatrix} x_1 \\ x_2 \\ L \\ K \end{bmatrix} = \begin{bmatrix} 1-a_{11} & -a_{12} & -w & 0 \\ -a_{21} & 1-a_{22} & 0 & 0 \\ -l_1 & -l_2 & 1 & 0 \\ -k_1 & -k_2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}.$$

If the solution makes economic sense, the model determines the levels of production and of use of the factors compatible with given real wages and investments. The values taken on by L e K may be, and in general are, different from the values of full employment.

After fixing (or determining) real wages, the price system becomes:

$$\begin{bmatrix} 1-a_{11}-wl_1 & -a_{21} & -k_1 \\ -a_{12}-wl_2 & 1-a_{22} & -k_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ rp_2 \end{bmatrix} = \mathbf{0}.$$

This is a homogeneous linear system of two equations in three unknowns. We need to fix one of the unknowns. It is appropriate to fix the price of one commodity, chosen as *numéraire*, which in this case may be the second commodity. If we set $p_2 = 1$, the price system becomes:

$$\begin{bmatrix} 1-a_{11}-wl_1 & -k_1 \\ -a_{12}-wl_2 & -k_2 \end{bmatrix} \begin{bmatrix} p_1 \\ r \end{bmatrix} = \begin{bmatrix} a_{21} \\ a_{22}-1 \end{bmatrix}.$$

The system may be solved as follows:

$$\begin{bmatrix} p_1 \\ r \end{bmatrix} = \begin{bmatrix} 1-a_{11}-wl_1 & -k_1 \\ -a_{12}-wl_2 & -k_2 \end{bmatrix}^{-1} \begin{bmatrix} a_{21} \\ a_{22}-1 \end{bmatrix}$$

thus determining the price of the first commodity and the profit rate which make sense, obviously, only on condition that the result is non-negative.

5. The open model with respect to consumption: production lag and the classical wage fund theory

The classical approach to economics represents the workings of the economic system as a circular process. Production takes place by using goods which were produced in an earlier round. These are used as raw materials and as means of subsistence for the workers. At the end of the production process, if the system is 'viable', there is a surplus equal to the quantity produced over and above the quantity used as a means of production.

In order to close the input-output model using the classical framework, two characteristic aspects should be taken into account [Morishima, 1989]:

- the existence of a time span necessary for the production of consumer goods, typically the year covered by one full cycle of agricultural production (hypothesis of *production lag*);

- the limits imposed by the fixed quantity of consumer goods produced and left over in the previous period, which the workers can purchase with the wages they receive (*wage fund* theory).

These two hypotheses change the input-output table quite significantly, which therefore becomes:

$p_1 a_{11} x_1$	$p_1 a_{12} x_2$	$p_1 C$	$p_1 I_1$	$p_1 x_1$
$p_2 a_{21} x_1$	$p_2 a_{22} x_2$		$p_2 I_2$	$p_2 x_2$
$p_1 w l_1 x_1$	$p_1 w l_2 x_2$			$p_1 w L$
$p_2 r k_1 x_1 + rA$	$p_2 r k_2 x_2$			$p_2 r K + rA$
$p_1 x_1$	$p_2 x_2$	$p_1 w L$	$p_2 r K + rA$	

.

Let us analyse the rows and columns. The first row refers to the production of consumer goods. One part of the production ($a_{11}x_1 + a_{12}x_2$) is re-employed in the production process, another part (C) is used to take the wage fund back to the previous level for use in the next period of production; while the last part (I_1) goes to increase the wage fund for the next period, so that a larger number of workers may be employed or real wages may be increased the following year. The decision to increase the wage fund is made by the capitalists, who set aside a proportion of the profits for this purpose.

The second row concerns the production of capital goods. The capitalists use up the profits, except for those needed to increase the wage fund, for new investments I_2. There is therefore a trade-off between I_1 and I_2; as a result, the process of accumulation takes place at the expense of employment and real wages.

The third row shows the number of workers and total wages, while the fourth shows total profits and the use of capital.

Since the production of consumer goods covers a period of time equal to a year, businessmen have to provide beforehand the means of production necessary for the period ahead. They receive interest on the money they make available, which is calculated at the same rate as the profit rate. The value of the advance is equal to:

$$A = p_1 a_{11} x_1 + p_2 a_{21} x_1 + p_1 w l_1 x_1 + p_2 r k_1 x_1$$

and thus the interest (profit) is equal to rA.

No advance is required for the production of capital goods, as this production is assumed to be instantaneous.

The quantity and price equations are as follows:

$$x_1 = a_{11}x_1 + a_{12}x_2 + C + I_1$$
$$x_2 = a_{21}x_1 + a_{22}x_2 + I_2$$
$$L = l_1 x_1 + l_2 x_2$$
$$K = k_1 x_1 + k_2 x_2$$

$$\begin{aligned} p_1 &= (1+r)(a_{11}p_1 + a_{21}p_2 + wl_1p_1 + rk_1p_2) \\ p_2 &= a_{12}p_1 + a_{22}p_2 + wl_2p_1 + rk_2p_2 \end{aligned}.$$

Written in matrix form, the input-output model may be represented as follows:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -1 & 0 & 0 & 0 \\ -a_{21} & 1-a_{22} & 0 & -1 & 0 & 0 \\ -l_1 & -l_2 & 0 & 0 & 1 & 0 \\ -k_1 & -k_2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ I_1 \\ I_2 \\ L \\ K \end{bmatrix} = \begin{bmatrix} C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-a_{11} & -a_{21} & -l_1 & -a_{21}-k_1 & -a_{11} & -l_1 & -k_1 \\ -a_{12} & 1-a_{22} & -l_2 & -k_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ wp_1 \\ rp_2 \\ rp_1 \\ wrp_1 \\ rrp_2 \end{bmatrix} = \mathbf{0}\,.$$

The quantity and price systems appear to be independent but are not really so, since the real wages (w) and the level of employment (L) are linked to the (*wage fund*) relationship:

$$C = wL\,.$$

As to the price system, we may fix a distributive variable, for example the profit rate, and the price of a commodity chosen as *numéraire*, for example $p_2 = 1$. This yields:

$$\begin{bmatrix} 1-a_{11}-a_{11}r & -l_1-l_1r \\ a_{12} & l_2 \end{bmatrix} \begin{bmatrix} p_1 \\ wp_1 \end{bmatrix} = \begin{bmatrix} a_{21}+k_1r+a_{21}r+k_1r^2 \\ 1-a_{22}-k_2r \end{bmatrix}$$

with the solution:

$$\begin{bmatrix} p_1 \\ wp_1 \end{bmatrix} = \begin{bmatrix} 1 - a_{11} - a_{11}r & -l_1 - l_1 r \\ a_{12} & l_2 \end{bmatrix}^{-1} \begin{bmatrix} a_{21} + k_1 r + a_{21} r + k_1 r^2 \\ 1 - a_{22} - k_2 r \end{bmatrix}.$$

The value of w can be calculated immediately by dividing wp_1 by p_1. The solution makes sense economically only if the result is not negative.

If we now move on to the quantity system, we observe that the variable L is now determinate and is equal to:

$$L = \frac{C}{w}.$$

Moreover, the second equation is in fact an identity, and can be eliminated. The quantity system may be reduced to:

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & -1 & 0 \\ l_1 & l_2 & 0 & 0 \\ -k_1 & -k_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ I_1 \\ K \end{bmatrix} = \begin{bmatrix} C \\ L \\ 0 \end{bmatrix}.$$

The quantity system offers one more degree of freedom and the value of one of the four variables x_1, x_2, I_1 or K may be fixed arbitrarily. Let us fix the value of I_1. If we make this choice the classical version of the input-output model is ‘open’ with respect to consumption, in that the final demand for consumer goods is completely autonomous.

The quantity system becomes:

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & 0 \\ l_1 & l_2 & 0 \\ -k_1 & -k_2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ K \end{bmatrix} = \begin{bmatrix} C + I_1 \\ L \\ 0 \end{bmatrix}$$

with the solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ K \end{bmatrix} = \begin{bmatrix} 1-a_{11} & -a_{12} & 0 \\ l_1 & l_2 & 0 \\ -k_1 & -k_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} C+I_1 \\ L \\ 0 \end{bmatrix}.$$

As we know the value of sectoral production, the new capital investments are equal to:

$$I_2 = x_2 - a_{21}x_1 - a_{22}x_2 .$$

The solutions are economically acceptable only if they are not negative.

6. Conclusions

In the preceding paragraphs we analysed the relationship between classical, marginalist and Keynesian economics and the Leontief model.

Leontief analysis does not deal with the paradigms of traditional economics. It is concerned pragmatically only with the problems of the impact of certain exogenous variables on the economic system in general, and on the production system in particular [Rose and Mierninyk, 1989]. Indeed, in its ‘open’ version, in which consumption and investments (and, in dual terms, wages and profits) play the role of exogenous variables, the model is ‘neutral’ with respect to traditional frames of reference or, if we prefer, it constitutes one particular simplification of these.

If we wish to ‘close’ the model, that is, if we consider the feedback of the levels of production on consumption and investments (and, in dual terms, of prices on wages and profits), we have to choose a theory of consumption and a theory of investment. At this point the Leontief model takes on characteristics that may align it variously, depending on whether we stand in the classical, the marginalist or the Keynesian camp.

In particular, we have seen how the classical wage fund assumption leads to an input-output model which is ‘closed’ with respect to investments, but ‘open’ with respect to consumption. The marginalist acceptance of Say's Law leads to a completely ‘closed’ model, with respect to both consumption and investments, while the non-acceptance of Say's Law lends a Keynesian-type connotation to the model, which thus becomes ‘closed’ with respect to consumption, but ‘open’ with respect to investments.

The problem of 'closure' highlights the influence of the different analytical approaches on the basic structure of the model itself.

We should point out, first of all, that in the first edition of his work [Leontief, 1941], Leontief presented input-output analysis as a model that was 'closed with respect to final demand', in which families were assimilated to any productive sector. Families represent industry $(n+1)$, which requires final goods as an input, and which provides work services as an output. This closure, in which final demand and value added are treated as residual quantities, is purely an accounting closure. It differs from the closures presented in the previous paragraphs in at least two ways: 1) the final demand is not disaggregated into its principal components; 2) theories which justify the dependence of consumption and investments on levels of production are not examined.

In the second edition of *The Structure of the American Economy* [Leontief, 1951], Leontief makes explicit reference to an 'open' model, in which final demand is autonomous. He quotes Keynes and realises that all practical applications of the model are aimed at assessing the impact of variations in final demand on the economic system. Multi-sectoral analysis develops the Keynesian multiplier principle in more detail.

If the Leontief model can be defined as Keynesian in its performance, its structure appears, at first sight, to be marginalist. Leontief accepts, on the one hand, the Walrasian simplification of fixed coefficients of production [Walras, 1874] (a practical hypothesis, but not a constituent part of the model, given that this is also fully compatible with the possibility of there being variable coefficients, as shown by Carter and Petri, [1989]). On the other hand, he applies Gustav Cassel's idea of deriving economic relations from empirical observation, rather than deducing them from the principle of maximum utility [Cassel, 1932; Arrow and Debreu, 1954]. The

Leontief model is one of general economic equilibrium, and its prototype can only be Leon Walras' mathematical model.

It may be argued, however, that the classical part of the Leontief model goes beyond Keynesian and marginalist connotations and suggests a concept of classical economic theory as an open-ended analytical framework.

The Leontief model is a circular model of production [Leontief, 1928], ‘in striking contrast to the view presented by modern theory, of a one-way avenue that leads from *Factors of production* to *Consumption goods*’ [Sraffa, 1960]. The idea of analysing in detail the relations between the various parts of an economic system may be traced back to François Quesnay and his *Tableau économique* [Quesnay, 1758]; while Karl Marx took it up again in his multi-sector reproduction model**.** In the best tradition of classical economics, the undoubtedly ambitious aim of Leontief analysis is to study the economic reality in all its complexity, accepting ‘a compromise between the unrestricted generalities of purely theoretical reasoning and the practical limitations of empirical fact finding’ [Leontief, 1941]. This is arguably a way to avoid widening the gap between ‘theory’ and ‘facts’ and to avert the danger of elegant theorising (irrelevant for practical purposes) or of empirical applications unsupported by an adequate theoretical framework.

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